AFOSR-TR-97-0640

REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1. AGENCY USE ONLY *(Leave Blank)*	2. REPORT DATE	3. REPORT TYPE AND DATES COVERED
	11/13/97	Final Technical Report (9/15/94 - 9/14/97)

4. TITLE AND SUBTITLE
Effects of Texture on the Plastic Anisotropy of Orthorhombic Sheets of Cubic Metals: A Group-Theoretic Analysis

5. FUNDING NUMBERS
G - F49620-94-1-0393

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9. SPONSORING / MONITORING AGENCY NAME(S) AND ADDRESS(ES)
AFOSR/NM
110 Duncan Avenue, Room B115
Bolling AFB, DC 20332-8080

10. SPONSORING / MONITORING AGENCY REPORT NUMBER

11. SUPPLEMENTARY NOTES

19971204 179

12a. DISTRIBUTION / AVAILABILITY STATEMENT
Approved for public release; distribution unlimited.

12b. DISTRIBUTION CODE

13. ABSTRACT *(Maximum 200 words)*

A plastic potential with seven texture coefficients, which shows explicitly how crystallographic texture would affect the plastic flow of sheet metals (e.g., aluminum) in metal-forming processes was developed. Formulae relating plastic strain ratios (r-values) of sheet metals to their texture coefficients, as predicted by this plastic potential, were derived. The derivation of this new plastic potential was based physically on the principle of material frame-indifference and mathematically on the theory of group representations and theory of invariants. The group-theoretic method described herein could be carried further if derivation of a plastic potential with higher texture coefficients should become desirable.

14. SUBJECT TERMS
anisotropic plasticity, metal-forming, sheet metals, r-values, crystallographic texture

15. NUMBER OF PAGES
17

16. PRICE CODE

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
U	U	U	UL

NSN 7540-01-280-5500

Standard Form 298 (Rev. 2-89)
Prescribed by ANSI Std. Z39-1
298-102

Executive Summary

A plastic potential showing explicitly how crystallographic texture would affect the plastic anisotropy of sheets of cubic metals (e.g., aluminum) was developed. This plastic potential is of the form

$$\begin{aligned} 2f(\boldsymbol{\sigma}, w) &= \frac{1}{Y_o^2}(\frac{3}{2}\operatorname{tr}\boldsymbol{\sigma}^2 + \alpha \operatorname{tr}\boldsymbol{\sigma}^3 + \beta\, \boldsymbol{\sigma} \cdot \boldsymbol{\Phi}(W_{400}, W_{420}, W_{440})[\boldsymbol{\sigma}] \\ &\quad + \gamma_1\, \boldsymbol{\sigma} \cdot \boldsymbol{\Psi}(W_{400}, W_{420}, W_{440})[\boldsymbol{\sigma}, \boldsymbol{\sigma}] \\ &\quad + \gamma_2\, \boldsymbol{\sigma} \cdot \Theta(W_{600}, W_{620}, W_{640}, W_{660})[\boldsymbol{\sigma}, \boldsymbol{\sigma}]); \end{aligned}$$

here $\boldsymbol{\sigma}$ is the deviatoric stress, and w is the orientation distribution function with expansion coefficients W_{lmn}; $\boldsymbol{\Phi}$ and Θ are explictly determined fourth-order and sixth-order harmonic tensors, respectively; $\boldsymbol{\Psi}$ is a sixth-order tensor expressed explicitly in terms of $\boldsymbol{\Phi}$; $Y_o, \alpha, \beta, \gamma_1$ and γ_2 are material constants. Formulae relating plastic strain ratios (r-values) of sheet metals to their texture coefficients, as predicted by this plastic potential were derived. The derivation of the new plastic potential was based physically on the principle of material frame-indifference and mathematically on the theory of group representations and theory of invariants. The group-theoretic method developed could be carried further if derivation of a plastic potential with higher texture coefficients (say, including those with $l = 8$) should become desirable.

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Publications Resulting from this Project

1. C.-S. Man, "On the correlation of elastic and plastic anisotropy in sheet metals", *Journal of Elasticity* **39**, 165–173 (1995).

2. C.-S. Man, "Correlation of earing and texture in weakly orthotropic sheets of cubic metals", in *Review of Progress in Quantitative Nondestructive Evaluation*, Vol. 14, D.O. Thompson and D.E. Chimenti (eds.), Plenum Press, New York, 1995, pp. 1923–1930.

3. C.-S. Man, and Q. Ao, “Plastic strain ratio and texture coefficients in orthotropic sheets of cubic metals”, in *Review of Progress in Quantitative Nondestructive Evaluation*, Volume 15, D.O. Thompson and D.E. Chimenti (eds.), Plenum Press, New York, 1996, pp. 1353–1360.

4. C.-S. Man, “On the constitutive equations of some weakly-textured materials”, to appear in *Arch. Rational Mech. Anal.*

5. C.-S. Man, “Effects of texture on plastic anisotropy in sheet metals”, to appear in *Nondestructive Characterization of Materials VIII*, R.E. Green (ed.), Plenum Press, New York.

Effects of Texture on the Plastic Anisotropy of Orthorhombic Sheets of Cubic Metals: A Group-Theoretic Analysis

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1 Introduction

Crystallographic texture determines the mechanical anisotropy of sheet metals and thus strongly influences their formability. For an alloy with a given composition, crystallographic texture is determined by the processing sequence employed to produce the end product. By suitably controlling thermomechanical processing parameters, a desirable blending of texture components could be obtained. The capability to monitor crystallographic texture on-line at various stages of the processing sequence is a prerequisite for controlling the texture of the end product.

Ultrasonic techniques offer a fast and inexpensive means for obtaining information on the texture of sheet metals on-line. By measuring in effect the texture coefficient W_{400}, ultrasonics has been used [1, 2] successfully for monitoring the average plastic strain ratio $\bar{r}$ of steel sheets. Efforts [3, 4] have also been made to use ultrasonics to characterize textures in aluminum alloys. As is well known, ultrasonics can deliver only very limited information on crystallographic texture. For the applications at hand, however, we are interested really not in crystallographic texture per se, but in its effect on formability of the sheet metal in question. Hence the bottom line is whether the information on texture that we could obtain from ultrasonics would suffice for our purpose. In this regard the success story on steel sheets serves as a good example. Earlier studies [5, 6] suggest that the texture coefficients W_{400}, W_{420}, W_{440} would specify, to good approximation, the plastic anisotropy of cold-rolled steel sheets. Through the efforts of many researchers, this observation has ultimately led to the successful application of ultrasonics for on-line monitoring of the formability parameter $\bar{r}$ in steel sheets.

As the plastic flow of a sheet metal in forming operations is determined by its plastic potential, which is often taken as identical to the yield function, it is natural to ask how crystallographic texture, as described by the orientation distribution function (ODF) w, would affect the yield function. Back in 1948 Hill [7] introduced a class of quadratic yield functions for describing the orthotropic plasticity of sheet metals. Hill's quadratic class of yield functions was widely adopted in the fifties and sixties

for modelling the plastic anisotropy of steel sheets. On the assumption that the anisotropic part of the yield function depends linearly on the texture coefficients W_{lmn} (which should be an adequate assumption if the sheet in question is weakly textured), it has been proved [8] recently that the principle of material frame-indifference in continuum mechanics entails the following theorem: For orthorhombic aggregates of cubic crystallites, the anisotropic part of any yield function in Hill's quadratic class can depend on w only in the three texture coefficients W_{400}, W_{420}, W_{440}; moreover, this dependence is explicitly determined up to a material-dependent multiplicative factor. The preceding theorem leads to an hitherto unnoticed prediction, which is borne out by experimental data on low-carbon steel sheets.[9, 10]

Hill's quadratic yield functions, however, are inadequate [11, 12] for describing the plastic behavior of aluminum. This finding has prompted research efforts to develop non-quadratic yield functions. Hill [13] himself has lately introduced a "user-friendly theory" by adding cubic terms to his 1948 quadratic. On the other hand, a recent work of Wagner and Lücke [14] suggests that a yield function with seven texture coefficients (W_{4m0} for $m = 0, 2, 4$, and W_{6m0} for $m = 0, 2, 4, 6$) might suffice for characterizing the plastic anisotropy of aluminum alloys.

In our project we consider only plastic flow under loading conditions. Complications regarding yielding, unloading, and reloading are not of our concern. All our discussions will be focused on only one constitutive function, namely, the plastic potential. While it is common to identify the plastic potential with the yield function (as Hill did), this association is unnecessary for our work and in what follows we shall refrain from making this extraneous assumption. Henceforth we shall use the symbol f to denote the plastic potential, and we make no commitment to any theory on the phenomena of yielding, unloading, and reloading.

For a given w, let us regard the plastic potential f as a smooth function of the deviatoric stress $\boldsymbol{\sigma}$ and expand f by Taylor's formula at $\boldsymbol{\sigma} = \mathbf{0}$. We may delete the constant term as it has no effect on the plastic flow, which is determined by the derivative of f with respect to $\boldsymbol{\sigma}$. The term linear in $\boldsymbol{\sigma}$ also drops out as a result of the presumed material symmetry. If we truncate the Taylor expansion of f at the quadratic term, we obtain the class of Hill's quadratic plastic potentials. Clearly a natural generalizaion of Hill's quadratic class would result if we truncate the expansion at a higher-order term.

Motivated by the work of Hill [13] and of Wagner and Lücke [14] mentioned above, in this project we truncated the Taylor expansion of f at the cubic term. We restricted our attention to weakly-textured orthorhombic sheets of cubic crystallites, and we assumed that the anisotropic part of f depends linearly on the texture coefficients W_{lmn} ($l \geq 1$). Extending our earlier work [8] on Hill's quadratic yield functions, we appealed to the principle of material frame-indifference and, without going into any detailed micromechanical modelling, tried to delineate the dependence of f on the ODF w as explicitly as possible. The theory of group representations and theory

of invariants provided the appropriate tools for us to achieve this goal and obtained representation formula (24) which shows explicitly the effects of texture on f. It turns out that this truncated f depends on the ODF only in the aforementioned seven texture coefficients, and that its anisotropic part can be written as a sum of three terms, each of which is determined to within a material-dependent multiplicative constant. After we obtained formula (24), we applied it to obtain formulae which show in what way the r-values depend on the seven texture coefficients.

In this Final Technical Report, we summarize our main findings in Sections 3 and 4 below. In Section 2 we go over some mathematical preliminaries to prepare for the discussion in Section 3 on the derivation of formula (24).

2 Decomposing a Tensor into its Irreducible Parts

Let V be the translation space of the three-dimensional Euclidean space, and let $T^{(r)}V$ be the space of r-th order tensors on V. A rotation $\boldsymbol{Q}$ on V induces a linear transformation $\boldsymbol{Q}^{\otimes r}$ on $T^{(r)}V$, and the map $\boldsymbol{Q} \mapsto \boldsymbol{Q}^{\otimes r}$ defines [15] a linear representation of the rotation group SO(3) on $T^{(r)}V$. Let Z be a subspace of $T^{(r)}V$ invariant under $\boldsymbol{Q}^{\otimes r}$ for each $\boldsymbol{Q}$, and let $\boldsymbol{Q}^{\otimes r}|Z$ be the restriction of $\boldsymbol{Q}^{\otimes r}$ on Z. Then $\boldsymbol{Q} \mapsto \boldsymbol{Q}^{\otimes r}|Z$ defines a linear representation of the rotation group on Z. We refer to these representations of SO(3) on tensor spaces as tensor representations. Tensor representations of the rotation group are, in general, not irreducible.

The rotation group can be parametrized by pairs $(\omega, \boldsymbol{n})$; for a rotation $\boldsymbol{R}(\omega, \boldsymbol{n})$, $0 \leq \omega \leq \pi$ gives the angle of rotation about an axis specified by the unit vector $\boldsymbol{n}$; it is understood that $\boldsymbol{R}(\pi, \boldsymbol{n}) = \boldsymbol{R}(\pi, -\boldsymbol{n})$ when $\omega = \pi$. The rotation group has a complete set of irreducible unitary representations D_l ($l = 0, 1, 2, ...$) of dimension $2l + 1$, with characters

$$\chi_l(\boldsymbol{R}(\omega, \boldsymbol{n})) = \chi_l(\omega) = \frac{\sin(l + \frac{1}{2})\omega}{\sin \frac{1}{2}\omega}. \tag{1}$$

Let $\chi(\omega)$ be the character of the representation $\boldsymbol{Q} \mapsto \boldsymbol{Q}^{\otimes r}|Z$, which decomposes into a direct sum

$$m_0 D_0 + m_1 D_1 + \cdots + m_r D_r, \tag{2}$$

with the multiplicity m_k ($k = 0, 1, ..., r$) given by the formula

$$m_k = \frac{2}{\pi} \int_0^\pi \chi(\omega) \overline{\chi_k(\omega)} \sin^2 \frac{\omega}{2} d\omega, \tag{3}$$

where $\overline{\chi_k}$ denotes the complex conjugate of χ_k; here $\overline{\chi_k} = \chi_k$ because χ_k is real.

In what follows we shall be particularly interested in the subspaces $[V^2]_0$, $[[V^2]_0^2]$, and $[[V^2]_0^3]$ of $T^{(2)}V$, $T^{(4)}V$, and $T^{(6)}V$, respectively. Here we have followed a system

of notation advocated by Jahn [16] and Sirotin [17, 18]: $[V^2]_0$ denotes the space of traceless, symmetric second-order tensors, $[[V^2]_0^2]$ the symmetric square of $[V^2]_0$ (i.e., the symmetrized tensor product of $[V^2]_0$ and $[V^2]_0$), and $[[V^2]_0^3]$ the symmetric cube of $[V^2]_0$. The characters χ_a, χ_b and χ_c of the representations $\boldsymbol{Q} \mapsto \boldsymbol{Q}^{\otimes 2}|[V^2]_0$, $\boldsymbol{Q} \mapsto \boldsymbol{Q}^{\otimes 4}|[[V^2]_0^2]$, and $\boldsymbol{Q} \mapsto \boldsymbol{Q}^{\otimes 6}|[[V^2]_0^3]$ can be easily found [16, 19] to be

$$\begin{aligned}
\chi_a &= 1 + 2\cos\omega + 2\cos 2\omega, && (4)\\
\chi_b &= 3 + 4\cos\omega + 4\cos 2\omega + 2\cos 3\omega + 2\cos 4\omega, && (5)\\
\chi_c &= 5 + 8\cos\omega + 8\cos 2\omega + 6\cos 3\omega + 4\cos 4\omega + 2\cos 5\omega + 2\cos 6\omega, && (6)
\end{aligned}$$

respectively. Following usual practice, we use the symbols $[V^2]_0$, ..., also to denote the corresponding representations $\boldsymbol{Q} \mapsto \boldsymbol{Q}^{\otimes 2}|[V^2]_0$, ..., respectively. From Eqs. (1), (3) and (4)–(6), we deduce that the tensor representations in question decompose into their irreducible parts as follows:[1]

$$\begin{aligned}
[V^2]_0 &= D_2, && (7)\\
[[V^2]_0^2] &= D_0 + D_2 + D_4, && (8)\\
[[V^2]_0^3] &= D_0 + D_2 + D_3 + D_4 + D_6. && (9)
\end{aligned}$$

Let us use Eq. (8) to illustrate the meaning of these decompositions. Let $\boldsymbol{C} \in [[V^2]_0^2]$; it can be taken as a fourth-order tensor which enjoys the major symmetry and maps $[V^2]_0$ (i.e., the space of traceless, symmetric second-order tensors) into $[V^2]_0$. Decomposition (8) asserts that $\boldsymbol{C}$ can be written as the direct sum of three tensors, which are in a 1-dimensional, 5-dimensional, and 9-dimensional subspace of $[[V^2]_0^2]$, respectively; the restrictions of the representations $\boldsymbol{Q} \mapsto \boldsymbol{Q}^{\otimes 4}$ to these subspaces are equivalent to the irreducible representations D_0, D_2, and D_4, respectively.

In what follows we shall consider classes of material tensors $\boldsymbol{Z} \in Z \subset T^{(r)}V$ pertaining to some textured polycrystalline aggregate. The subspace Z will be invariant under $\boldsymbol{Q}^{\otimes r}$ for each $\boldsymbol{Q} \in \mathrm{SO}(3)$, and the material tensors $\boldsymbol{Z}$ will be functions of the ODF w. For weakly-textured aggregates, we shall take $\boldsymbol{Z}$ to be linear in w. Moreover, the tensor functions $\boldsymbol{Z}(w)$ that we shall consider will naturally satisfy a constraint of the following form:

$$\boldsymbol{Z}(\mathcal{T}_{\mathbf{Q}}(w))[\boldsymbol{Q}\boldsymbol{v}_1, ..., \boldsymbol{Q}\boldsymbol{v}_r] = \boldsymbol{Z}(w)[\boldsymbol{v}_1, ..., \boldsymbol{v}_r] \tag{10}$$

for each $\boldsymbol{Q} \in \mathrm{SO}(3)$, and any $\boldsymbol{v}_1, ..., \boldsymbol{v}_r \in V$; here, with $\boldsymbol{Q}^T$ standing for the transpose of $\boldsymbol{Q}$, $\mathcal{T}_{\mathbf{Q}}(w)$ denotes the rotated ODF defined by

$$\mathcal{T}_{\mathbf{Q}}(w)(\boldsymbol{R}) = w(\boldsymbol{Q}^T\boldsymbol{R}) \tag{11}$$

[1]The problem of decomposing a tensor into its irreducible parts under the rotation group was studied by Sirotin [20]. He advocated a two-step approach, in which the decomposition is carried out first [21] with respect to the general linear group GL(3), and then [22] with respect to the rotation group. Here we follow a more direct method using characters. Sirotin did not consider the classes of tensors which concern us here.

for each $\boldsymbol{R} \in \mathrm{SO}(3)$. Under the preceding constraint and the linearity assumption, it has been proved [8] that the r-th order tensor $\boldsymbol{Z}(w)$ can depend only on those texture coefficients W_{lmn} with $1 \leq l \leq r$. Let us resolve $\boldsymbol{Z}(w)$ into its irreducible parts (cf. expression (2)), i.e., writing $\boldsymbol{Z}$ as a direct sum of tensors ${}^{(j)}\boldsymbol{Z}_k$ ($j = 1, ..., m_k$ when the multiplicity $m_k \neq 0$; $k = 0, 1, ..., r$); ${}^{(j)}\boldsymbol{Z}_k$ lies in a subspace corresponding to the irreducible representation D_k. A more refined argument [19] similar to the one given in [8] shows that ${}^{(j)}\boldsymbol{Z}_k$ can depend only on those texture coefficients W_{lmn} with $l = k$. Irreducible parts corresponding to the representation D_0 are the isotropic parts of $\boldsymbol{Z}$.

The meaning of the decompositions (7)–(9) in the present context should now be apparent. Consider $\boldsymbol{C} \in [[V^2]_0^2]$ again for instance. Decomposition (8) implies that we can write $\boldsymbol{C}$ in terms of its irreducible parts as

$$\boldsymbol{C} = \boldsymbol{C}_0 + \boldsymbol{C}_2 + \boldsymbol{C}_4; \tag{12}$$

here $\boldsymbol{C}_0$ is the isotropic part of $\boldsymbol{C}$; $\boldsymbol{C}_2$ can depend on the ODF w only in the coefficients W_{2mn}, and $\boldsymbol{C}_4$ only in W_{4mn}. As we shall see in the next section, when $m_k = 1$, the relation of an anisotropic irreducible part to the texture coefficients is determined up to an arbitrary multiplicative (material) constant. Thus, with decomposition (12) in hand, the effects of texture on the anisotropy of $\boldsymbol{C}$ will be determined up to two material constants.

Before we proceed further, let us record another fact for later use. The tensor space $T^{(r)}V$ contains [23] only one invariant subspace $Z_h^{(r)}$ of dimension $2r+1$ for which the representation $\boldsymbol{Q} \mapsto \boldsymbol{Q}^{\otimes r}|Z_h^{(r)}$ is equivalent to D_r. Moreover, $Z_h^{(r)}$ is the subspace of harmonic tensors [24, 25] in $T^{(r)}V$, i.e., those which are totally symmetric and traceless. If we denote the tensors in $T^{(r)}V$ by $\boldsymbol{H}$ and their components by $H_{i_1 i_2 \cdots i_r}$, then the harmonic tensors satisfy

$$H_{i_1 i_2 \cdots i_r} = H_{i_{\tau(1)} i_{\tau(2)} \cdots i_{\tau(r)}} \tag{13}$$

for each permutation τ of $\{1, 2, ..., r\}$, and

$$\mathrm{tr}_{j,k} \boldsymbol{H} = \boldsymbol{0} \tag{14}$$

for any pair of distinct indices j and k.

3 A Plastic Potential Which Shows Explicitly the Effects of Seven Texture Coefficients

Let $[V^2]_0$ be the set of traceless symmetric second-order tensors, and let w be the ODF. We consider plastic potentials f with w and the deviatoric stress $\boldsymbol{\sigma}$ as independent variables. The principle of material frame-indifference [26] dictates that f must satisfy [8] the identity

$$f(\boldsymbol{Q}\boldsymbol{\sigma}\boldsymbol{Q}^T, \mathcal{T}_{\mathbf{Q}}(w)) = f(\boldsymbol{\sigma}, w) \tag{15}$$

for each rotation $\boldsymbol{Q}$, each $\boldsymbol{\sigma}$ in $[V^2]_0$, and each ODF w; recall that $\boldsymbol{Q}^T$ stands for the transpose of $\boldsymbol{Q}$, and $\mathcal{T}_{\mathbf{Q}}(w)$ denotes the rotated ODF defined by $\mathcal{T}_{\mathbf{Q}}(w)(\boldsymbol{R}) = w(\boldsymbol{Q}^T\boldsymbol{R})$ for each rotation $\boldsymbol{R}$.

Henceforth we restrict our attention to weakly-textured orthorhombic sheets of cubic metals. Throughout this report we shall always use a spatial coordinate system whose coordinate axes fall on the axes of orthorhombic symmetry of the polycrystalline aggregate, and we shall choose a reference orientation for the constituting crystallites such that the three 4-fold axes of cubic symmetry of the reference orientation agree with the coordinate axes of the chosen spatial coordinate system. With this choice of coordinate system and reference crystallite orientation, which we shall henceforth refer to as the standard setting, orthorhombic texture symmetry and cubic crystal symmetry [27, 28, 29] dictate that for $1 \le l \le 7$, all $W_{lmn} = 0$ except for some with $l = 4$ or 6. For $l = 4$ and $l = 6$, there are three and four independent texture coefficients, respectively, and we may and will pick W_{4m0} $(m = 0, 2, 4)$ and W_{6m0} $(m = 0, 2, 4, 6)$ to fill that role.

For the present discussion we assume that f is smooth and is linear in w. For a fixed w, we express f in its Taylor expansion at $\boldsymbol{\sigma} = \mathbf{0}$ and truncate the expansion at the cubic term. Thus we have

$$f(\boldsymbol{\sigma}, w) = f(\mathbf{0}, w) + D_1 f(\mathbf{0}, w)[\boldsymbol{\sigma}] + \frac{1}{2!}D_1^2 f(\mathbf{0}, w)[\boldsymbol{\sigma}, \boldsymbol{\sigma}] + \frac{1}{3!}D_1^3 f(\mathbf{0}, w)[\boldsymbol{\sigma}, \boldsymbol{\sigma}, \boldsymbol{\sigma}]. \tag{16}$$

As the plastic flow will be determined by the derivative of f with respect to $\boldsymbol{\sigma}$, we may simply drop the term $f(\mathbf{0}, w)$. We may rewrite the other three terms in Eq. (16) in terms of three tensors $\boldsymbol{B}(w)$, $\boldsymbol{C}(w)$ and $\boldsymbol{D}(w)$ in $[V^2]_0$, $[[V^2]_0^2]$ and $[[V^2]_0^3]$, respectively, as follows:

$$D_1 f(\mathbf{0}, w)[\boldsymbol{\sigma}] = \boldsymbol{\sigma} \cdot \boldsymbol{B}(w), \tag{17}$$

$$\frac{1}{2!}D_1^2 f(\mathbf{0}, w)[\boldsymbol{\sigma}, \boldsymbol{\sigma}] = \boldsymbol{\sigma} \cdot \boldsymbol{C}(w)[\boldsymbol{\sigma}], \tag{18}$$

$$\frac{1}{3!}D_1^3 f(\mathbf{0}, w)[\boldsymbol{\sigma}, \boldsymbol{\sigma}, \boldsymbol{\sigma}] = \boldsymbol{\sigma} \cdot \boldsymbol{D}(w)[\boldsymbol{\sigma}, \boldsymbol{\sigma}]. \tag{19}$$

Note that $\boldsymbol{C}(w)$ is a fourth-order tensor which enjoys the major symmetry, and $\boldsymbol{D}(w)$ is a sixth-order tensor which satisfies

$$\boldsymbol{\sigma}_1 \cdot \boldsymbol{D}(w)[\boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3] = \boldsymbol{\sigma}_{\tau(1)} \cdot \boldsymbol{D}(w)[\boldsymbol{\sigma}_{\tau(2)}, \boldsymbol{\sigma}_{\tau(3)}] \tag{20}$$

for each permutation τ of $\{1, 2, 3\}$.

The constraint (15) on f dictates that the tensors $\boldsymbol{B}$, $\boldsymbol{C}$ and $\boldsymbol{D}$ satisfy

$$\begin{aligned} \boldsymbol{B}(\mathcal{T}_{\mathbf{Q}}(w)) &= \boldsymbol{Q}\boldsymbol{B}(w)\boldsymbol{Q}^T, && (21) \\ \boldsymbol{C}(\mathcal{T}_{\mathbf{Q}}(w))[\boldsymbol{Q}\boldsymbol{\sigma}\boldsymbol{Q}^T] &= \boldsymbol{Q}\boldsymbol{C}(w)[\boldsymbol{\sigma}]\boldsymbol{Q}^T, && (22) \\ \boldsymbol{D}(\mathcal{T}_{\mathbf{Q}}(w))[\boldsymbol{Q}\boldsymbol{\sigma}\boldsymbol{Q}^T, \boldsymbol{Q}\boldsymbol{\sigma}\boldsymbol{Q}^T] &= \boldsymbol{Q}\boldsymbol{D}(w)[\boldsymbol{\sigma}, \boldsymbol{\sigma}]\boldsymbol{Q}^T, && (23) \end{aligned}$$

for each $\boldsymbol{\sigma}$ in $[V^2]_0$, each ODF w, and each $\boldsymbol{Q}$ in SO(3). Under the present assumptions, we conclude from the decompositions (7)–(9) what follows:

1. Since all W_{2mn} are zero, $\boldsymbol{B} = \mathbf{0}$.

2. Corresponding to the terms D_0 and D_4 in decomposition (8), $\boldsymbol{C}$ can be written as the direct sum of an isotropic term and a term depending only on W_{400}, W_{420} and W_{440}.

3. Since all W_{3mn} are also zero, according to the decomposition (9), $\boldsymbol{D}$ is the direct sum of three terms, the first being isotropic, the second depending only on W_{400}, W_{420} and W_{440}, and the third depending only on W_{600}, W_{620}, W_{640} and W_{660}.

Representation formulae for isotropic fourth-order and sixth-order tensors are well known. The isotropic terms in question are proportional to $\mathrm{tr}\boldsymbol{\sigma}^2$ and to $\mathrm{tr}\boldsymbol{\sigma}^3$, respectively. A glance at Eqs. (22) and (23) reveals that these constraints can at best determine each irreducible part of $\boldsymbol{C}$ or $\boldsymbol{D}$ up to an arbitrary multiplicative constant. Thus we may express the plastic potential f in the form

$$\begin{aligned} 2f(\boldsymbol{\sigma}, w) \;=\; \frac{1}{Y_o^2}\Big(&\frac{3}{2}\,\mathrm{tr}\boldsymbol{\sigma}^2 + \alpha\,\mathrm{tr}\boldsymbol{\sigma}^3 + \beta\,\boldsymbol{\sigma}\cdot\boldsymbol{\Phi}(W_{400}, W_{420}, W_{440})[\boldsymbol{\sigma}] \\ &+\gamma_1\,\boldsymbol{\sigma}\cdot\boldsymbol{\Psi}(W_{400}, W_{420}, W_{440})[\boldsymbol{\sigma}, \boldsymbol{\sigma}] \\ &+\gamma_2\,\boldsymbol{\sigma}\cdot\boldsymbol{\Theta}(W_{600}, W_{620}, W_{640}, W_{660})[\boldsymbol{\sigma}, \boldsymbol{\sigma}]\Big); \end{aligned} \tag{24}$$

here $\boldsymbol{\Phi} \in [[V^2]_0^2]$; $\boldsymbol{\Psi}, \boldsymbol{\Theta} \in [[V^2]_0^3]$; $Y_o, \alpha, \beta, \gamma_1$ and γ_2 are material constants. Since $\boldsymbol{\Phi}$ corresponds to the irreducible representation D_4 in $T^{(4)}V$, it is harmonic and is thence totally symmetric and traceless. Similarly, the sixth-order tensor $\boldsymbol{\Theta}$ is totally symmetric and traceless.

The explicit dependence of the harmonic tensors $\boldsymbol{\Phi}$ and $\boldsymbol{\Theta}$ on the texture coefficients can be determined [19] by appealing to Schur's lemma and to the constraints (22) and (23). (The same results can also be obtained [8, 30] directly from (22) and

(23) by brute force.) As for $\mathbf{\Phi}(w)$, because of the orthorhombic texture and cubic crystal symmetry, there are only three independent components:

$$\begin{aligned}
\Phi_{1122} &= W_{400} - \sqrt{70}W_{440}, \\
\Phi_{1133} &= -4W_{400} + 2\sqrt{10}W_{420}, \\
\Phi_{2233} &= -4W_{400} - 2\sqrt{10}W_{420},
\end{aligned} \tag{25}$$

where we have arbitrarily chosen a normalization constant so that the preceding equations look simplest. All other non-trivial components of $\mathbf{\Phi}$ can be obtained through the total symmetry of $\mathbf{\Phi}$, except for the following three, which follow from the traceless condition:

$$\begin{aligned}
\Phi_{1111} &= -(\Phi_{1122} + \Phi_{1133}) \\
&= 3W_{400} - 2\sqrt{10}W_{420} + \sqrt{70}W_{440}, \\
\Phi_{2222} &= -(\Phi_{1122} + \Phi_{2233}) \\
&= 3W_{400} + 2\sqrt{10}W_{420} + \sqrt{70}W_{440}, \\
\Phi_{3333} &= -(\Phi_{1133} + \Phi_{2233}) \\
&= 8W_{400}.
\end{aligned} \tag{26}$$

The harmonic tensor $\mathbf{\Theta}(w)$ has seven independent components:

$$\begin{aligned}
\Theta_{111122} &= -W_{600} + \frac{\sqrt{105}}{15}W_{620} + \sqrt{14}W_{640} - \sqrt{231}W_{660}, \\
\Theta_{111133} &= 6W_{600} - \frac{16\sqrt{105}}{15}W_{620} + 2\sqrt{14}W_{640}, \\
\Theta_{222211} &= -W_{600} - \frac{\sqrt{105}}{15}W_{620} + \sqrt{14}W_{640} + \sqrt{231}W_{660}, \\
\Theta_{222233} &= 6W_{600} + \frac{16\sqrt{105}}{15}W_{620} + 2\sqrt{14}W_{640}, \\
\Theta_{333311} &= -8W_{600} + \frac{16\sqrt{105}}{15}W_{620}, \\
\Theta_{333322} &= -8W_{600} - \frac{16\sqrt{105}}{15}W_{620}, \\
\Theta_{112233} &= 2W_{600} - 2\sqrt{14}W_{640},
\end{aligned} \tag{27}$$

where we have again arbitrarily chosen a normalization constant. All the other non-trivial components of $\mathbf{\Theta}$ can be determined from the preceding seven through the total symmetry of $\mathbf{\Theta}$, except for the following three, which are obtained from the

traceless condition:

$$\begin{aligned}
\Theta_{111111} &= -(\Theta_{111122} + \Theta_{111133}) \\
&= -5W_{600} + \sqrt{105}W_{620} - 3\sqrt{14}W_{640} + \sqrt{231}W_{660}, \\
\Theta_{222222} &= -(\Theta_{222211} + \Theta_{222233}) \\
&= -5W_{600} - \sqrt{105}W_{620} - 3\sqrt{14}W_{640} - \sqrt{231}W_{660}, \\
\Theta_{333333} &= -(\Theta_{333311} + \Theta_{333322}) \\
&= 16W_{600}.
\end{aligned} \tag{28}$$

Finally, let us consider the sixth-order tensor $\mathbf{\Psi}(w)$, which resides in a 9-dimensional subspace $Z \subset [V^2]_0^3]$ such that the representation $\boldsymbol{Q} \mapsto \boldsymbol{Q}^{\otimes 6}|Z$ is equivalent to the representation $\boldsymbol{Q} \mapsto \boldsymbol{Q}^{\otimes 4}|Z_h^{(4)}$, where $Z_h^{(4)}$ denotes the subspace of harmonic tensors in $T^{(4)}V$. Thus there is a linear mapping (i.e., a tenth-order tensor) $\boldsymbol{M} : T^{(4)}V \to T^{(6)}V$ such that the restriction of $\boldsymbol{M}$ to $Z_h^{(4)}$ is an isomorphism of $Z_h^{(4)}$ onto Z. For a given w, let us denote a generic element in $Z_h^{(4)}$ by $\mathbf{\Phi}(w)$ and its image under $\boldsymbol{M}$ by $\mathbf{\Psi}(w)$, i.e.,

$$\Psi_{ijklmn} = M_{ijklmnabcd}\Phi_{abcd}. \tag{29}$$

(Here, to avoid introducing new symbols, we have abused the language and used the symbols $\mathbf{\Phi}(w)$ and $\mathbf{\Psi}(w)$ in a more general context. The general relation between $\mathbf{\Phi}(w)$ and $\mathbf{\Psi}(w)$, which we shall derive presently, remains valid in the special context at issue.) The constraints (22) and (23) on $\mathbf{\Phi}(w)$ and $\mathbf{\Psi}(w)$, respectively, imply that $\boldsymbol{M}$ is a tenth-order isotropic tensor. By a theorem ([31], p. 260) in the theory of invariants, we can express the isotropic tensor $\boldsymbol{M}$ as a linear combination of terms each of which can be obtained from

$$\delta_{ij}\delta_{kl}\delta_{mn}\delta_{ab}\delta_{cd}$$

by permuting the indices. After taking into account the conditions that $\mathbf{\Phi}$ is totally symmetric and traceless, that $\mathbf{\Psi} : [V^2]_0 \times [V^2]_0 \to [V^2]_0$, and that $\mathbf{\Psi}$ enjoys the minor symmetries and the symmetry specifed by Eq. (20), we obtain the following formula for the components of $\mathbf{\Psi}$, which determines $\mathbf{\Psi}$ to within an arbitrary multiplicative constant:

$$\begin{aligned}
\Psi_{ijklmn} = \; & -8(\delta_{ij}\Phi_{klmn} + \delta_{kl}\Phi_{ijmn} + \delta_{mn}\Phi_{ijkl}) + 3(\delta_{ik}\Phi_{jlmn} + \delta_{jk}\Phi_{ilmn} \\
& +\delta_{il}\Phi_{jkmn} + \delta_{jl}\Phi_{ikmn} + \delta_{mk}\Phi_{nlij} + \delta_{nk}\Phi_{mlij} + \delta_{ml}\Phi_{nkij} \\
& +\delta_{nl}\Phi_{mkij} + \delta_{im}\Phi_{jnkl} + \delta_{jm}\Phi_{inkl} + \delta_{in}\Phi_{jmkl} + \delta_{jn}\Phi_{imkl}).
\end{aligned} \tag{30}$$

In our present context we have already determined $\mathbf{\Phi}(w)$ up to a multiplicative constant. Indeed the components of $\mathbf{\Phi}(w)$ under the standard setting and a particular

choice of normalization constant are given in Eqs. (25) and (26) above. Substituting these explicit expressions into (30) defines the tensor $\boldsymbol{\Psi}$ in Eq. (24). Twenty components of $\boldsymbol{\Psi}$ are displayed explicitly in a table given in the Appendix. Any non-trivial component not given there can be obtained from those displayed by appealing to the symmetries in the indices of Ψ_{ijklmn}.

Similar to the parameters in Hill's quadratic yield functions, the texture coefficients in Eq. (24) should be interpreted as "parameters characteristic of the current state of anisotropy" ([32], p. 318). Evolution of texture during deformations is outside the scope of our present investigation.

Besides the texture coefficients, which reflect the influence of texture, the plastic potential f given in (24) contains five material constants Y_0, α, β, γ_1 and γ_2.[2] This plastic potential will fall in Hill's quadratic class if we drop all the cubic terms in (24), i.e., by setting $\alpha = 0$ and $\gamma_1 = \gamma_2 = 0$.

4 Application: Relations of R-Values to Texture

Let a homogeneous sample sheet of some cubic metal be given. We assume that the given sheet is orthorhombic, with its axes of sample symmetry in the rolling direction (RD), the transverse direction (TD), and the normal direction (ND) of the sheet, respectively. We choose a spatial Cartesian coordinate system such that the 1-, 2-, and 3-direction agrees with RD, TD, and ND, respectively. We adopt a reference orientation for the cubic crystallites such that the three four-fold axes of cubic symmetry of the reference orientation fall on the spatial coordinate axes.

Let $r(\theta)$ be the plastic strain ratio in the direction in the plane of the sheet which makes an angle θ with the 1-direction of the chosen spatial coordinate system. Let

$$\overline{r} = \frac{1}{4}(r(0) + r(\pi/2) + 2r(\pi/4)), \tag{31}$$

$$\triangle r = \frac{1}{2}(r(0) + r(\pi/2) - 2r(\pi/4)), \tag{32}$$

$$\delta r = r(0) - r(\pi/2). \tag{33}$$

Using the function f given by Eq. (24) as the plastic potential in the flow rule, we deduce that the r-values defined in Eqs. (31)–(33) are given, correct to terms linear

[2]The constants β, γ_1 and γ_2 here are different from their namesakes in our earlier papers [8, 9, 10, 33, 34], because we have chosen for the tensors $\boldsymbol{\Phi}$, $\boldsymbol{\Psi}$, and $\boldsymbol{\Theta}$ new normalization constants which give the formulae above a simpler look.

in the texture coefficients, by the formulae

$$\bar{r} = 1 - aW_{400} - bW_{600}, \tag{34}$$

$$\triangle r = \frac{2\sqrt{70}}{5} aW_{440} - \frac{2\sqrt{14}}{7} bW_{640}, \tag{35}$$

$$\delta r = \frac{4\sqrt{10}}{5} aW_{420} + b\left(\frac{34\sqrt{105}}{105} W_{620} - \frac{2\sqrt{231}}{7} W_{660}\right), \tag{36}$$

where

$$a = 10\beta + 20\sigma_0\gamma_1, \qquad b = -7\sigma_0\gamma_2, \tag{37}$$

and σ_0 is the isotropic limit of the uniaxial flow stress when the r-values are measured.

If we set $\gamma_1 = \gamma_2 = 0$ in Eqs. (34)–(36), then these equations will reduce to those [9, 10, 33] that follow from a plastic potential in Hill's quadratic class.

5 Closing Remarks

The conventional method of measuring directional dependence of ultrasonic wave speeds will directly deliver only the texture coefficients W_{400}, W_{420} and W_{440}. If the plastic potential given in Eq. (24) could adequately model the plastic flow of aluminum alloys under loading conditions, evaluation of the texture coefficients W_{6m0} for $m = 0, 2, 4, 6$ should be on the agenda of researchers who strive to develop measurement systems for on-line monitoring of formability parameters of aluminum alloys. In this regard the work of Sakata et al. [35] on cold-rolled steel sheets might serve as a possible point of departure.

The group-theoretic method presented above, in principle, can be carried further if it should become desirable to keep even higher-order terms in the Taylor expansion of the plastic potential f. We must keep theory and practice in the proper perspective, however, as the more terms we keep, the more material constants will result in the expression for f, which will soon render the expression a theoretical curiosity with no practical applications.

References

[1] M. Hirao, H. Fukuoka, K. Fujisawa, and R. Murayama. On-line measurement of steel sheet $\bar{r}$-value using magnetostrictive-type EMAT. *J. Nondest. Eval.*, 12:27–32, 1993.

[2] K. Kawashima, T. Hyoguchi, and T. Akagi. On-line measurement of plastic strain ratio of steel sheet using resonance mode EMAT. *J. Nondest. Eval.*, 12:71–77, 1993.

[3] A.J. Anderson, R.B. Thompson, R. Bolingbroke, and J.H. Root. Ultrasonic characterization of rolling and recrystallization textures in aluminum. *Textures and Microstructures*, 26:39–58, 1996.

[4] W. Lu, D. Hughes, and S. Min. Texture measurement using emat and laser ultrasonics. In *ICOTOM-11*, pages 134–139. 1996.

[5] C.A. Stickels and P.R. Mould. The use of Young's modulus for predicting plastic-strain ratio of low-carbon steel sheets. *Metall. Trans.*, 1:1303–1312, 1970.

[6] G.J. Davies, D.J. Goodwill, and J.S. Kallend. Elastic and plastic anisotropy in sheets of cubic metals. *Metall. Trans.*, 3:1627–1631, 1972.

[7] R. Hill. A theory of yielding and plastic flow of anisotropic metals. *Proc. Roy. Soc.*, A 193:281–297, 1948.

[8] C.-S. Man. On the constitutive equations of some weakly-textured materials. To appear in *Arch. Rational Mech. Anal.*

[9] C.-S. Man. Elastic compliance and Hill's quadratic yield function for weakly orthotropic sheets of cubic metals. *Metall. Mater. Trans.*, 25A:2835–2837, 1994.

[10] C.-S. Man and Q. Ao. Plastic strain ratio and texture coefficients in orthotropic sheets of cubic metals. In D.O. Thompson and D.E. Chimenti, editors, *Review of Progress in Quantitative Nondestructive Evaluation*, volume 15B, pages 1353–1360. Plenum, New York, 1996.

[11] J. Woodthorpe and R. Pearce. The anomalous behaviour of aluminum sheet under balanced biaxial tension. *Int. J. mech. Sci.*, 12:341–347, 1970.

[12] F. Barlat. Crystallographic texture, anisotropic yield surfaces and forming limits of sheet metals. *Materials Science and Engineering*, 91:55–72, 1987.

[13] R. Hill. A user-friendly theory of orthotropic plasticity in sheet metals. *Int. J. Mech. Sci.*, 35:19–25, 1993.

[14] P. Wagner and K. Lücke. Quantitative correlation of texture and earing in Al-alloys. *Materials Science Forum*, 157–162:2043–2048, 1994.

[15] W. Miller, Jr. *Symmetry Groups and Their Applications.* Academic Press, New York, 1972.

[16] H.A. Jahn. Note on the Bhagavantam-Suryanarayana method of enumerating the physical constants of crystals. *Acta Cryst.*, 2:30–33, 1949.

[17] Yu.I. Sirotin. Group tensor spaces. *Sov. Phys. Crystallogr.*, 5:157–165, 1960.

[18] Yu.I. Sirotin and M.P. Shaskolskaya. *Fundamentals of Crystal Physics.* Mir, Moscow, 1982.

[19] C.-S. Man. In preparation.

[20] Yu.I. Sirotin. Decomposition of material tensors into irreducible parts. *Sov. Phys. Crystallogr.*, 19:565–568, 1975.

[21] G.F. Smith. *Constitutive equations for anisotropic and isotropic materials.* North-Holland, Amsterdam, 1994.

[22] F.D. Murnaghan. On the decomposition of tensors by contraction. *Proc. Nat. Acad. Sci.*, 38:973–979, 1952.

[23] I.M. Gel'fand, R.A. Minlos, and Z.Ya. Shapiro. *Representations of the Rotation and Lorentz Groups and Their Representations.* MacMillan, New York, 1963.

[24] G. Backus. A geometric picture of anisotropic elastic tensors. *Reviews of Geophysics and Space Physics*, 8:633–671, 1970.

[25] R. Baerheim. Harmonic decomposition of the anisotropic elasticity tensor. *Q. Jl Mech. appl. Math.*, 46:391–418, 1993.

[26] C. Truesdell and W. Noll. *The Non-Linear Field Theories of Mechanics.* Springer, Berlin, second edition, 1992.

[27] R.-J. Roe. Description of crystallite orientation in polycrystalline materials. iii. general solution to pole figures. *J. Appl. Phys.*, 36:2024–2031, 1965.

[28] R.-J. Roe. Inversion of pole figures for materials having cubic crystal symmetry. *J. Appl. Phys.*, 37:2069–2072, 1966.

[29] P.R. Morris. Symmetry requirements on W_{lmn} with odd l for cubic crystal symmetry. *Textures and Microstructures*, 4:241–242, 1982.

[30] R. Paroni and C.-S. Man. Constitutive equations of elastic polycrystalline materials. In preparation.

[31] A.J.M. Spencer. Theory of invariants. In A.C. Eringen, editor, *Continuum Physics*, volume 1, pages 239–353. Academic Press, New York, 1971.

[32] R. Hill. *The Mathematical Theory of Plasticity.* Clarendon Press, Oxford, 1950.

[33] C.-S. Man. On the correlation of elastic and plastic anisotropy in sheet metals. *J. Elasticity*, 39:165–173, 1995.

[34] C.-S. Man. Effects of texture on plastic anisotropy in sheet metals. In R.E. Green, Jr., editor, *Nondestructive Characterization of Materials VIII.* Plenum, New York. To appear.

[35] K. Sakata, D. Daniel, and J.J. Jonas. Estimation of 4th and 6th order odf coefficients from elastic properties in cold rolled steel sheets. *Textures and Microstructures*, 11:41–56, 1989.

Appendix

A Table for the Components of Ψ

Every component Ψ_{ijklmn} of Ψ can be written as a linear combination of W_{400}, $\sqrt{10}W_{420}$ and $\sqrt{70}W_{440}$. The following table lists the coefficients of such linear combinations for 20 non-trivial components of Ψ. For instance, we have

$$\Psi_{111111} = 36W_{400} - 24\sqrt{10}W_{420} + 12\sqrt{70}W_{440}.$$

All the other non-trivial components of Ψ can be obtained from those displayed below by appealing to the minor symmetries of Ψ (i.e., $\Psi_{ijklmn} = \Psi_{jiklmn}$, etc.) and to the symmetry specified by Eq. (20) (i.e., $\Psi_{ijklmn} = \Psi_{klijmn}$, etc.).

	W_{400}	$\sqrt{10}W_{420}$	$\sqrt{70}W_{440}$
Ψ_{111111}	36	-24	12
Ψ_{111122}	-28	16	-4
Ψ_{111133}	-8	8	-8
Ψ_{222211}	-28	-16	-4
Ψ_{222222}	36	24	12
Ψ_{222233}	-8	-8	-8
Ψ_{333311}	-48	-8	0
Ψ_{333322}	-48	8	0
Ψ_{333333}	96	0	0
Ψ_{112233}	56	0	8
Ψ_{112323}	23	22	-3
Ψ_{113131}	-19	8	3
Ψ_{111212}	16	-6	-4
Ψ_{222323}	-19	-8	3
Ψ_{223131}	23	-22	-3
Ψ_{221212}	16	6	-4
Ψ_{332323}	-4	-14	0
Ψ_{333131}	-4	14	0
Ψ_{331212}	-32	0	8
Ψ_{233112}	-21	0	-3